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The Lee Fields Medal VII

TIME ALLOWED: UP TO TWO HOURS AND 15 MINUTES

TABLES AND CALCULATORS MAY BE USED. EACH QUESTION WORTH 10 MARKS.

1. Show that there are no strictly positive integers k, m, and n such that

$$3^{2m} - 2^{2n} = 7^k.$$

2. Let $a > 0$ be a real number. Calculate

$$\cdots\sqrt{\cdots\sqrt{\sqrt{\sqrt{a}}}},$$

where the number of square roots grows to infinity. Justify your answer.

3. Which real numbers are solutions of the equation:

$$(x+1)^3 - x(x+1)^2 = (x+1)^2?$$

4. Show that if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are distinct points on the curve

$$y = mx + c,$$

that the slope of PQ is equal to m.

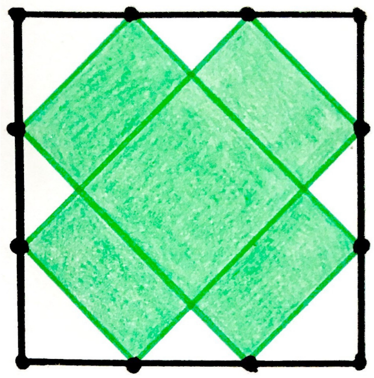
5. Suppose that a function f is given by an infinite sum:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

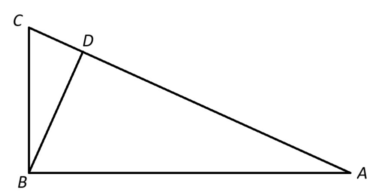
Assuming that term-by-term differentiation is possible for this infinite sum, given that $a_0 = 1$, find the general term a_n of a non-zero sequence of real numbers $(a_0, a_1, a_2, a_3, \dots)$ such that the derivative of f is equal to f itself:

$$f'(x) = f(x).$$

6. The dots are equally spaced. What fraction of the square is shaded?



7. Consider a triangle ΔABC with $\angle ABC$ a right-angle, and D a point on $[AC]$ such that $[BD]$ is perpendicular to $[AC]$:



Prove that

$$\sin^2(\angle BAC) = \frac{|CD|}{|AC|}.$$

8. Consider three decks of playing cards, one belonging to Alice, one belonging to Bob, and one belonging to a referee. Alice chooses, not at random, their 21 favourite cards from their deck. Bob chooses, again not at random, their favourite 36 cards from their deck. The referee takes one card at random from their own deck. Let A be the event that the referee chose one of Alice's favourite cards, and B the event that the referee chose one of Bob's favourite cards.

   (a) Find $\mathbb{P}[A]$, the probability that the referee chose one of Alice's favourite cards.

   (b) Find $\mathbb{P}[B]$.

   (c) Recalling that Alice and Bob did *not* choose their favourite cards *at random*, find the maximum possible value of $\mathbb{P}[A \cap B]$, the probability that the referee chose a card that was a favourite of BOTH Alice AND Bob.

   (d) Similarly, find the minimum possible value of $\mathbb{P}[A \cap B]$.

9. In rugby, teams T_1 and T_2 can score three points with a penalty or drop goal, five points with an unconverted try, and seven points with a converted try. If the order of scores is not important, how many different ways could a match end with both teams scoring 20 points?

   Note, T_1 with a drop goal and T_2 with a penalty is a 'different' 3-3 to the 3-3 where both T_1 and T_2 score a penalty.

10. There are three money boxes, labelled "Euro", "Sterling", and "Mixed". You cannot look inside a box, but can press a button on it to remove a coin from that box.

    One of the boxes has euro coins in it, another has sterling coins, and in the third a mixture of the two. But ALL three money boxes are labeled incorrectly. What is the minimum number of coins you can remove to work out which box is which?